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Economic growth, savings and income distribution

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Abstract

This paper develops a macroeconomic model of growth and income distribution. Starting from a simple framework it is found that over time economies develop through two cumulative processes, a short and a long run. The model raises the conjecture that, in contrast to the usual assumption of a savings and investment equality, savings and investments are in fact two different things. Relaxing this assumption has immediate consequences for the view on short and long run economic development and may also help to explain macroeconomically the phenomena of bubbles and crisis. Finally, income distribution is introduced to the double cumulative growth process to show the effects of changes in income distribution on aggregate growth.

Content

Introduction

In principle, economic processes should be simple. After all, almost all world population are not trained economists, do not know the economic models or the data and prior to the year 1776 economics as a science did not even exist. Still, before and after 1776 economies existed and developed, and even though they have become more sophisticated over time, the basic processes behind all this may have changed not too much. This paper is an attempt to analyse such processes that govern the growth of income and its distribution in an economy. Accordingly, the starting point of the paper is a simple macroeconomic setting, which, by thinking it through, develops into a number of theories on why economies grow and how income distribution and changes therein may affect this. As it turns out, even when starting with a very simple framework, economic processes become increasingly complex, especially if the dynamics over time are considered to be important, as it is done here. Additionally, the paper aimed at deriving these theories in a logically consistent manner and that is why extensive use of mathematics was made, which may reduce the readability.

What the paper ends up with is that over time economies might develop via two cumulative processes, one short and one long run. Thereby, the main and potentially only factors determining growth from a macroeconomic point of view are technology and investment. Furthermore, the paper raises the conjecture that economically savings are neither equal nor identical to investments, which has immediate consequences on the short and long run economic development. All this is found in an aggregate model and, by extending it, in a model including income distribution.

Hence, in the first part, a basic macroeconomic model is developed to which in a second step savings are introduced. After that income distribution is introduced to the basic model, which in a last step is than again enriched by some aspects of savings.

The basic model

Over time, a country's income can be regarded to develop through two cumulative processes, a short run and a long run cumulative process. In the long run, the income generated in one year, or any other defined time period, is the basis for the next year's income, which is itself the basis for the subsequent year's income and so on. Over the course of the years, this income may grow, decline or stay constant, depending on what is done with the income in the short run.

In the short run there are two simple possibilities: the income may be used or it may not be used. The general reason for income to exist is reproduction. Income is generated because we work, and we can survive and work because we generate income. The more productive we are, the higher will be our income and the better we survive. This does not exclude the possibility that income is not used. Part of the income may be destroyed by accident, another part may be saved for later use, or part of the income is not used because we produced much more than we actually need.

Still, excluding the destruction of income, there is some logic in the thought that the available income is used to the extent that it serves at least reproduction, i.e. being able to produce the same amount of income or more in the period to come, and only the part of income that goes beyond this will not immediately be used. From this it follows that the size of the income in the next period is determined by what is done with the income used.

For the sake of getting a logical model on how aggregate income develops both in the short and the long run, two distinct activities on how income may be used are identified. Firstly, part of the income may be used for reproduction purposes only, assuming that this enables to reproduce exactly the same amount of income in the next period. Secondly, current income may also be used in a way that it not only serves for reproduction but additionally for an increase in aggregate income. In analogy to the national accounting framework the first activity is labelled consumption and the second investment.

However the concept of 'investment' used here is more encompassing than the definition of investment in national accounting. One the one hand, in line with national accounting, it contains the creation of fixed assets, e.g. machines. By devoting a certain part of the income to such investment, this part firstly serves those constructing the fixed assets as a means for reproduction, while secondly through the newly created fixed assets additional income can be created. On the other hand, 'investment' also includes other activities, that instead of the production of fixed assets, entail an increase in aggregate income by increasing the level of (aggregate) productivity, e.g. through learning effects.

As in this context consuming a share of the income means that this part of the income is used up and not available for other use and just allows to reproduce the share of income consumed, short run growth of income thus depends on how much of the income is spent for investment activities and the extent to which these investments generate additional income.

As a matter of fact the additional income created by investment is causal for the short run cumulative process, as it may be used again for consumption and investment purposes. If it is totally used for consumption, the cumulative process ends, as all of the additional income is used up. If however part of it is, again, used for investment, this creates additional income, which again may be spent as consumption or investment expenditure and so on.

This cumulative process resembles a Keynesian multiplier process, where the original share of income devoted to investment is multiplied. However, the Keynesian multiplier assumes that an increase in investment leads to an initial increase in consumption expenditure, thus generating additional income, leading to additional consumption expenditures and so on. By contrast, the cumulative process assumes that consumption expenditure does not generate additional income, because the additional income is consumed. Rather it claims that the initial investment leads to subsequent investment expenditures and this increases aggregate income. An important point here is that for any consumption expenditure something (e.g. a good or service of the same value) is received in return. Thus, for a given income an increase in consumption may lead to a higher turnover, but not to a higher income. Only if this expenditure leads to the production of something new income would increase. But this corresponds to investment (in the broader sense) as defined above.

To formalise the short run cumulative process, an initial income Y_0 at point 0 in time is assumed, whereby Y_0 could be the result of a simple or sophisticated production process of the period before. Furthermore let Y_0 be spent on consumption C_0 and investment I_0. Hence:

$$Y_0 = C_0 + I_0 \tag{1}$$

Since both consumption and investment are based on initial income, they can be also given as a fraction of initial income, so that:

$$Y_0 = bY_0 + cY_0 \tag{2}$$

With $b = C_0/Y_0$, $c = I_0/Y_0$ and $c + b \leq 1$.

As noted above, investment expenditures create additional income and thus are the starting point of the short run cumulative process. The additional income is given by: cY_0/v, whereby v is the capital-output ratio, i.e. it states how much capital is needed to produce one unit of output (income). The lower v is, the more output can be produced with a given capital and investment level, and thus represents also a higher state of technology and/or productivity. It represents the level of technology applied in production and includes not only the technology embodied in the means of production, but also the sophistication of the work process and in a broader definition the skills and experience of the workforce, too.

For the sake of the argument, v is kept constant and set to 1, so that in the first round the additional income Y_0' because of investment expenditures is cY_0. This is again spent on consumption C_0' and investment I_0':

$$Y_0' = cY_0 = C_0' + I_0' \tag{3}$$

or as consumption and investment can be again expressed as a share of income:

$$cY_0 = cbY_0 + ccY_0 \tag{4}$$

Correspondingly the second round additional income is:

$$c^2Y_0 = c^2bY_0 + c^2cY_0 \tag{5}$$

And in the n-th round additional income is given by:

$$c^nY_0 = c^nbY_0 + c^ncY_0 \tag{6}$$

Correspondingly total income $Y_{i,0}^f$ at the end of the accumulation process, i.e. the sum of initial income plus the additional income generated by investment injections, is given by:

$$Y_{i,0}^f = Y_{i,0} + cY_{i,0} + c^2Y_{i,0} + \cdots + c^nY_{i,0} \tag{7}$$

Which for n going to infinity and c < 1 converges to:

$$Y_0^f = \frac{Y_{i,0}}{1-c} \tag{8}$$

Thus, aggregate income is the higher the lower the propensity to consume and the higher the propensity to invest is. If v is different from 1 aggregate income develops into:

$$Y_0^f = \frac{Y_0}{1 - c/v} \tag{9}$$

Correspondingly total consumption through this process is:

$$C_0^f = b\frac{Y_0}{1 - c/v} \tag{11}$$

Total investment is given by:

$$I_0^f = c\frac{Y_0}{1 - c/v} \tag{12}$$

As a side aspect, it is also possible to derive a 'quasi' investment multiplier from this model. Unlike the Keynesian multiplier, investment is not fully exogenous, because initial consumption C_0 and investment I_0 are financed out of initial income Y_0. Hence a one unit change a change in investment necessitates a corresponding change in consumption. Taking this into account the 'quasi' investment multiplier of the cumulative model can be approximated by (see the Appendix for the derivation):

$$\frac{dY}{dI} = \frac{1}{v(1-{}^{c}/_{v})^2} \tag{13}$$

Coming back, the final aggregate income of period 0 Y_0^f can be considered to be the initial income available in period 1, so that

$$Y_1 = Y_0^f \tag{14}$$

Assuming the same cumulative process being at work in period one, aggregate income at the end of this period is:

$$Y_1^f = Y_2 = \frac{Y_1}{1-{}^{c}/_{v}} = \frac{Y_0}{(1-{}^{c}/_{v})^2} \tag{15}$$

As the final income of period one is also the starting income of period 2 and so on, aggregate income (under constant c and v) for the n-th period is consequently given by:

$$Y_n^f = \frac{Y_{i,0}}{(1-{}^{c}/_{v})^n} \tag{16}$$

Thus aggregate income grows by the factor $1 - {}^{c}/_{v}$ from period to period. In terms of a growth rate this would correspond to a rate $g(Y)$ of:

$$g(Y) = \frac{c}{v-c} \tag{17}$$

So that for small values aggregate income for the n-th period can also be given by:

$$Y_n = Y_0 e^{\frac{c}{v-c}n} \tag{18}$$

Supply, demand and capacity utilisation

So far, the model implicitly assumed that all income is used either for consumption or investment and no income is saved. One immediate effect of this was that aggregate supply automatically equalled aggregate demand. Because of this, the model emphasised the demand side a bit stronger than the supply side, which basically was only defined via the capital output ratio v. Entering into an analysis of the effects of savings requires to be a bit more explicit about the supply side.

A convenient way to do this is using the concept of potential income Y^*, which formally is defined as:

$$Y^* = \frac{K}{v} \tag{19}$$

That is, potential income is a function of the size of the capital stock K and the capital-output ratio v. Corresponding to the broad notion of investment, as the capital stock is the accumulated sum of investments over time, K includes not only all physical capital used for production like machinery, buildings, infrastructure but also non-tangible capital like patents, know how etc., and in extremis even labour.

Accordingly, Y^* increases as the capital stock and the level of technology increase. The capital stock itself grows as new investment is added to the existing capital stock, while it decreases as the existing capital depreciates. More formally the change in the capital stock ΔK can be given as $\Delta K = I - dK$, where d is the amortisation rate of capital[1].

An eclectic approach to link aggregate demand with aggregate supply is to introduce to the concept of capacity utilisation. It expresses the extent to which actual income Y corresponds to potential income Y^*, or in other words how much of the income that optimally could be produced is actually demanded.

Formally capacity utilisation is expressed via the rate of capacity utilisation u:

$$Y = uY^* \tag{20}$$

with $u \leq 1$.

Given that in the widest sense the capital stock also includes labour, capacity utilisation not only measures the extent to which the physical capital stock is used, but also indicates the level of unemployment, so that a state of full capacity utilisation can be considered a state of full employment.

To analyse the relation between actual and potential income in more detail, the growth of potential income is derived through the growth of the capital stock (assuming technology and hence v to stay constant). By definition, the capital stock is the accumulated sum of investments over time. Hence, assuming the capital stock to be K_0 in period 0, it is K_1 in period 1 and given by K_0+I_0 and so forth for later periods. Formally, the capital stock in the n-th period is therefore given by:

$$K_n = K_0 + c\frac{Y_0}{1-{}^{c}/_{v}} + c\frac{Y_0}{(1-{}^{c}/_{v})^2} + \cdots + c\frac{Y_0}{(1-{}^{c}/_{v})^n} \tag{21}$$

where $c\,Y_0/(1-{}^{c}/_{v})^n$ is investment in period n (see equation 12). Transforming above equation it can also be given as:

$$K_n = K_0 + vY_0\frac{1-(1-{}^{c}/_{v})^n}{(1-{}^{c}/_{v})^n} \tag{22}$$

In the case of no investment, i.e. c = 0, the capital stock does not increase over time, but stays at K_0, while it increases the faster the higher c is, i.e. the higher share of investment out of income are[2].

Furthermore, for n being small, the capital stock will grow at a slower rate than demand side income[3]. If n grows large the growth rate of GDP and the capital stock will however converge. This is can be shown by writing the capital stock growth rate as:

$$\frac{K_n}{K_{n-1}} = \frac{(1-{}^{c}/_{v})^n K_0 - vY_0(1-{}^{c}/_{v})^n + vY_0}{(1-{}^{c}/_{v})^n K_0 - vY_0(1-{}^{c}/_{v})^n + vY_0(1-{}^{c}/_{v})} \tag{23}$$

[1] The remainder of the analysis disregards depreciation effects for simplicity reasons.

[2] For v = 1 and given that c = (1-b) this equation would simplify to: $K_n = K_0 + Y_0\frac{1-b^n}{b^n}$

[3] This is basically because the numerator $1-(1-{}^{c}/_{v})^n$ is smaller than 1, for n being not too large.

For n being large, this expression reduces to $1/(1-{}^{c}/_{v})$, which corresponds to the change in demand side income. Subtracting 1 in order to get the rate of growth consequently gives that in the limit:

$$g(K_n) = \frac{c}{v-c} \tag{24}$$

As far as the development of capacity utilisation u is concerned, it can be shown to develop according to the following path:

$$u_n = \frac{Y_n}{Y_n^*} = \frac{Y_0/(1-{}^{c}/_{v})^n}{\frac{1}{v}\left(K_0 + vY_0\frac{1-(1-{}^{c}/_{v})^n}{(1-{}^{c}/_{v})^n}\right)} \tag{25}$$

Here the numerator corresponds to the development of real income, while the denominator represents the development of potential income, as $u_n = Y_n/Y_n^*$. With some transformation and applying $u_0 = Y_0/K_0$, the path of capacity utilisation can also be given as:

$$u_n = \frac{u_0}{(1-{}^{c}/_{v})^n + u_0[1-(1-{}^{c}/_{v})^n]} \tag{26}$$

As n grows large, the denominator reduces to u_0, so that the whole expression gives 1.

$$u_n \underset{\infty}{\rightarrow} 1 \tag{27}$$

Hence capacity utilisation in the long run converges towards one, i.e. a state of full use of capacities – as long as there is investment (i.e. c > 0). Correspondingly, real income and potential income become identical. For this convergence to occur, the initial assumption has to be made that $Y_0^* > Y_0$. From the logic of the model so far, this is a situation, which actually cannot occur, because, as all income is used in one way or the other, potential income equals actual income right from the start and hence $u = 1$ at all times. Still it illustrates that, if there is a shock to actual or potential income, there is a natural tendency of both to equalise over time. More importantly, it helps understanding the effects of savings.

Savings

So far, the model was based on the assumption that a given income Y_0 is fully spent on either consumption or investment activities. However, an alternative assumption is that not all income is expended but part of it is saved for later use. This brings about some changes to the model. Most importantly, the main difference between expenditure and saving out of a given income is that the latter does not contribute to the formation of new income (at least not in the same period). In the short run, savings are not used for reproduction or the creation of additional income. They constitute the part of income that is reserved for future use.

Still, the saved part of income does not disappear (in the absence of depreciation). Rather it builds a stock of saved income, which may generate income in the periods to follow, as part of the savings stock is spent as consumption or investment expenditures. Hence, by assuming that aggregate income develops in a cumulative process over time, saving and investing occur simultaneously, but they are two distinctly different activities. As a consequence, savings cannot be considered to be equal to investments, neither in the short run nor in the long run[4].

In formal terms, initial income Y_0 is used for consumption, investment and saving, denoted by C_0, I_0 and S_0, respectively.

$$Y_0 = C_0 + I_0 + S_0 \tag{28}$$

As in the basic model above, the three elements can be expressed as shares of initial income. Thereby, b and c represent again the propensity to consume and to invest, while d is the savings rate, hence $d = S_0/Y_0$.

$$Y_0 = bY_0 + cY_0 + dY_0 \tag{29}$$

As all income is used for these three purposes, there has to be: $b + c + d = 1$.

Again, investment expenditures create additional income, thereby starting the short run cumulative process. Hence, first round additional income Y_0' is given by:

$$Y_0' = cY_0 = C_0' + I_0' + S_0' \tag{30}$$

or as consumption and investment can be again expressed as a share of income:

$$cY_0 = bcY_0 + ccY_0 + dcY_0 \tag{31}$$

Thereby, for simplicity reasons, a capital output ratio of v=1 is assumed at the moment. Correspondingly the n-th round additional income is given by:

$$c^nY_0 = bc^nY_0 + cc^nY_0 + dc^nY_0 \tag{32}$$

The crucial point is that the size of the aggregate income at the end of the period is exclusively determined by the part of income that constitutes expenditure, i.e. consumption and investment. This is because consumption is used for reproduction and investment for the creation of additional income (and reproduction), while in each round the amount saved is taken out of the cumulative process and enters into a stock of savings.

Consequently, the aggregate income at the end of the period, which is identical to the starting income of the next period Y_1 can be given in the long form as:

$$Y_1 = (b + c)Y_0 + (b + c)cY_0 + (b + c)c^2Y_0 + \cdots + (b + c)c^nY_0 \tag{33}$$

Using standard mathematics this expression reduces in the limit to:

[4] This is at odds with standard conventions that assume S = I. These conventions are based on the definition of income (for a closed economy without government) as a) Y = C+I and b) Y = C+S. This says that a) income is used for either consumption or investment and b) income that is not consumed is saved. From that the equality I = S follows directly. The misconception here is that in the end also investment expenditures contain consumption of income. For example, buying a machine for production is considered to be investment, yet it generates income for those producing the machine. This income may be consumed or again invested (setting of the cumulative process). Alternatively, the misconception might be more trivial, if saving is thought of as putting it on a bank account. Yet, as the bank usually lends this money out mostly to finance investments, these 'savings' are actually investments, if only indirectly. Hence, the equality S = I is in fact an identity, as such savings are nothing else than investments, and rather relate to how and by whom investments are financed. But they are not savings in the literal sense.

$$Y_1 = \frac{b+c}{1-c} Y_0 = \frac{1-d}{1-c} Y_0 \tag{34}$$

Accordingly, the accumulated savings in period 0 are given by:

$$S_0 = d \frac{1-d}{1-c} Y_0 \tag{35}$$

The development of aggregate income over time depends on the extent to which savings flow back into the economy, either in the form of investment or consumption. To demonstrate this, it is first assumed that there is no backflow of savings at all to the economy. In this case aggregate income simply develops according to:

$$Y_n = \left(\frac{1-d}{1-c}\right)^n Y_0 \tag{36}$$

Correspondingly, accumulated savings in the n-th period are given by:

$$S_n = Y_0 d(1-d) \frac{1-\left(\frac{1-d}{1-c}\right)^n}{d-c} \tag{37}$$

Alternatively, if there is a complete backflow of savings from one period to the next, i.e. the savings of period 0 enter as income in period 1, aggregate income is represented by:

$$Y_1^* = Y_1 + S_0 = bY_1^* + cY_1^* + dY_1^* \tag{38}$$

Whereby Y_1^* is the starting income of period 1, consisting of the income generated in period 0 and the complete savings of period 0, that now flow back into the economy. Using this it can be shown that the aggregate income for the n-th period Y_n is equal to:

$$Y_n = Y_0 \left(\frac{1-d}{1-c}\right)^n (1+d)^{n-1} \tag{39}$$

The total stock of savings amounts to:

$$S_n = Y_0 d \left(\frac{1-d}{1-c}\right)^n (1+d)^{n-1} \tag{40}$$

Hence, the stock of savings consists just of the savings of the last period, as all other savings re-entered the economy.

These two extreme cases represent the upper and lower bound for the development of aggregate income and the stock of savings over time. For any case, where the backflow of savings is incomplete, i.e. only a part of savings re-enter the economy, aggregate income and savings are between these bounds. In any case, as long as the savings rate is larger than 0, aggregate income develops slower than in the case of no savings (see APPENDIX).

Savings and capacity utilisation

The introduction of savings has, besides lowering the development path of aggregate income, also implications on capacity utilisation. This is firstly illustrated for the case of no backflows of savings. In this case, the capital stock can be shown to develop according to:

$$K_n = K_0 + c \frac{1-d}{1-c} Y_0 \left[1 + \frac{1-d}{1-c} + \left(\frac{1-d}{1-c}\right)^2 + \cdots + \left(\frac{1-d}{1-c}\right)^{n-1}\right] \tag{41}$$

In a shorter version this corresponds to

$$K_n = K_0 + c(1-d)Y_0 \frac{1-\left(\frac{1-d}{1-c}\right)^n}{d-c} \tag{42}$$

Assuming v=1 for the moment, capacity utilisation in the n-th period u_n can be given as: $u_n = Y_n/K_n$. Substituting for Y_n and K_n results in a less attractive representation of u_n:

$$u_n = \frac{\left(\frac{1-d}{1-c}\right)^n Y_0}{K_0 + c(1-d)Y_0 \frac{1-\left(\frac{1-d}{1-c}\right)^n}{d-c}} \tag{43}$$

In the limit however, it can be shown that this expression reduces to:

$$u_n = \frac{c-d}{c(1-d)} \tag{44}$$

Hence over time capacity utilisation converges to a fixed value lower than one, i.e. capacity is not fully utilised. It is easily checked that in the absence of savings, i.e. d=0, there is full capacity utilisation in the long run.

Similar is done for the case of complete backflows of savings. Over time the capital stock develops into:

$$K_n = K_0 + c(1-d)Y_0 \frac{1-\left(\frac{1-d}{1-c}\right)^n (1+d)^n}{d^2-c} \tag{45}$$

From this, long run capacity utilisation is given by

$$u_n = \frac{Y_0\left(\frac{1-d}{1-c}\right)^n (1+d)^{n-1}}{K_0 + c(1-d)Y_0 \frac{1-\left(\frac{1-d}{1-c}\right)^n (1+d)^n}{d^2-c}} \tag{46}$$

Also this nasty expression reduces to a simple expression:

$$u_n = \frac{c-d^2}{c-cd^2} \tag{47}$$

In the case of complete backflows long run capacity utilisation is lower than one, too, though it is higher as in the case of no backflows. In both cases, capacity utilisation is decreasing in d, as an increased savings rate tends to lower u_n. By contrast, ceteris paribus, capacity utilisation increases as the investment propensity c increases.

Including technology

Relaxing the assumption of v=1, i.e. setting it to a value of above 1, just as in the basic model above, tends to lower the development path of aggregate income. This is shown for the income in the n-th period Y_0 in the case of no backflows:

$$Y_n = Y_0 \left(\frac{1-d}{1-{}^{c}/_{v}}\right)^n \tag{48}$$

Correspondingly, the rate of capacity utilisation in the case of no savings backflows is given by:

$$u_n = \frac{{}^{c}/_{v} - d}{c(1-d)} \tag{49}$$

In the case of full backflows u_n transforms into:

$$u_n = \frac{{}^{c}/_{v} - d^2}{c - cd^2} \tag{50}$$

Abandoning the savings-investment identity as they are two different things and including this into a model of economic growth, delivers a simple and straightforward explanation, why, from a macroeconomic point of view, economies may grow at stable pace in the presence of underutilised capacities. Given the broader definition of capital and hence the assumption that capacity utilisation also reflects the extent of unemployment in an economy, the existence of savings may also explain the permanent presence of unemployment despite economic growth. Additionally, because consumption, investment and savings are connected to each other, as their shares in income add up to one, this model provides a direct link to Okun's law. First of all it shows that for stable technology levels and investment and savings propensities an economy may grow without changes in capacity utilisation and unemployment. Only if investment (under constant technology) or also consumption increases at the cost of savings, economic growth and capacity utilisation will be higher.

In a way, the inclusion of savings also links supply side theory with demand side theories. If savings are zero, i.e. all available income is used for consumption or investment, supply side theories have a certain point, though in the double cumulative model the distinction between aggregate supply and aggregate demand becomes blurry, because everything supplied is demanded, but this again generates supply. However, as soon as there are savings, the demand side becomes important, because, as it has been shown, if part of the income supplied is not brought to use, capacity utilisation and economic growth are lower than they could be.

The model gives also rise to the conjecture that the existence of savings might be, from a macroeconomic perspective, the reason for the recurrent phenomena of economic bubbles and crises. The model has shown that if the savings rate is higher than the investment rate aggregate income declines just as capacity utilisation (including technology makes the system even more sensible). So, in the build-up of a crisis what may happen is that the savings rate increases up to a point where it is higher than the investment rate, so that the economy enters into a decline. This may the case for housing or stock market bubbles, where more and more income is 'invested' into an existing stock of capital. The suspicion is that, from a macroeconomic point of view, these 'investments' are effectively savings as the income is not used productively. Yet, being a conjecture, it is clear that the mechanisms behind a crisis are much more complex. Especially, it needs to regard money and prices which are completely absent in this model.

Income distribution

From a macroeconomic point of view, one way to approach the link between economic growth and income is via the demand side aggregate income Y_i, dividing it into the sum of wages (W_i), i.e. the employees' remuneration, and profits (P_i), i.e. the income out of capital ownership, with i denoting country i.

$$Y = W + P \tag{51}$$

To analyse the relationship between income distribution and growth, some basic relationships between profit and wages on the one side and the size of aggregate income and capacity utilisation on the other are to be developed.

Starting with the profit rate, its potential size is determined via the following considerations. Assuming that all income is in the form of profits, i.e. there are no wages paid, aggregate income is equal to profits, i.e. $Y = P$. Hence, the equality between income and profits can also be given as:

$$Y = rK \tag{52}$$

Since real income Yi is equal to potential income times capacity utilisation, $Y = uY^*$, and potential income itself is determined by the ratio of the capital stock to the capital output ratio, $Y^* = K/v$, the upper limit of the profit rate is given by:

$$r^{max} = \frac{u}{v} \tag{53}$$

From this it follows, that the maximum profit rate is on the one hand determined by the level of technology. Thus, the higher the level of technology is, which corresponds to a lower v as more output can be produced with the same amount of capital, they higher can be the profit rate. On the other hand it also depends on the level of capacity utilisation, as idle capital produces no returns.

Without specifying an explicit production function and given that not all income is in the form of profits, the range of the realised profit rate r^K is then given by:

$$0 \leq r^K \leq \frac{u}{v} \tag{54}$$

whereby if r^K equals zero would correspond to a situation where all income is in the form of wages.

For analytical purposes it is made use of the fact that profits can be either defined via the profit rate times the capital stock or as a share of aggregate income p. Assuming for simplicity a capital output ratio of $v = 1$ and putting aggregate income in terms of potential income gives:

$$r^K K = P = puK \tag{55}$$

Dividing by K and rearranging then gives:

$$\frac{r^K}{u} = p \tag{56}$$

Hence for $u = 1$ it must hold that

$$p = r^K \tag{57}$$

Making use of the above fact that r^K varies with capacity utilisation, and reaches its maximum if capacity is fully utilised $(u = 1)$, the profit share p may be regarded as representation of this maximum profit rate. Hence, defining the maximum profit rate as r, allows to rewrite above equation (56) as $r^K = ur$, so that as an effect profits can be defined as:

$$r^K K = P = rY \tag{58}$$

Correspondingly, wages W can then be defined as:$W = (1 - r)Y$, so that total income is:

$$Y = (1 - r)Y + rY \tag{59}$$

For the analysis of the relationship between aggregate income growth and income distribution it is assumed that the employees' income Y_E not only consists of wages. Rather, because they also invest part of their income and therefore are also capital owners, they also receive a share of profits, which adds to their total income. Contrastingly, the income of 'pure' capital owners Y_C are profits only and depends on the share they have in the total economy's capital. For analytical purposes the capitalists' share is denoted by α, and the employees share in the economy's capital by $(1 - \alpha)$. Hence initial aggregate income Y^0 is the sum of employees' initial income Y_E^0 and capitalists' initial income Y_C^0:

$$Y^0 = Y_E^0 + Y_C^0 \tag{60}$$

This can also be expressed in the following way:

$$Y^0 = (1 - r)Y^0 + (1 - \alpha)rY^0 + \alpha rY^0 \tag{61}$$

The first two terms on the r.h.s. of the equation are employees initial income Y_E^0, while the third terms corresponds to capitalists' income Y_C^0.

Like above, the development of aggregate income is subject to a short and long run cumulative process, as the initial income of employees and capitalists can be either consumed or invested, with the latter being the driving force behind the cumulative processes. Yet, from an analytical point of view, the introduction of distributional aspects makes these a bit more complex, as all, the initial employees' wages, employees' profits and capitalists' profits are starting points of a cumulative process.

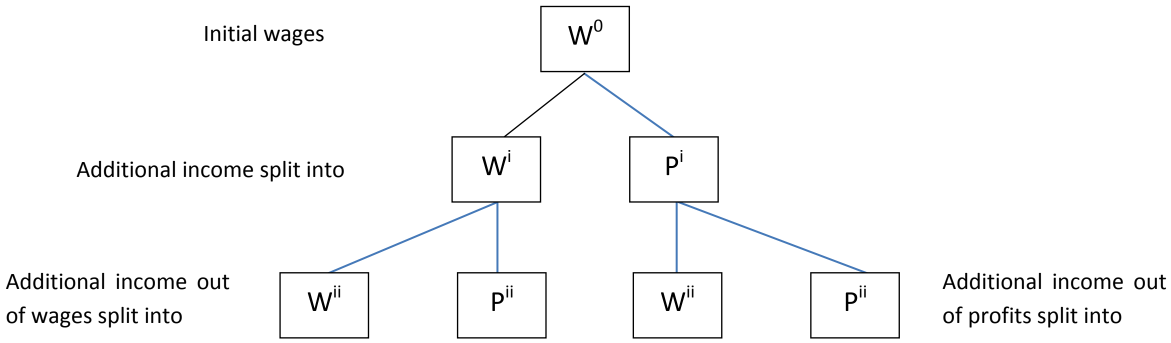
Regarding initial employees' wages $Y_{E,W}^0$, they are used for consumption and investment, according to the employees propensities to consumer b_E and to invest c_E. Hence:

$$Y_{E,W}^0 = (1 - r)Y^0 = b_E(1 - r)Y^0 + c_E(1 - r)Y^0 \tag{62}$$

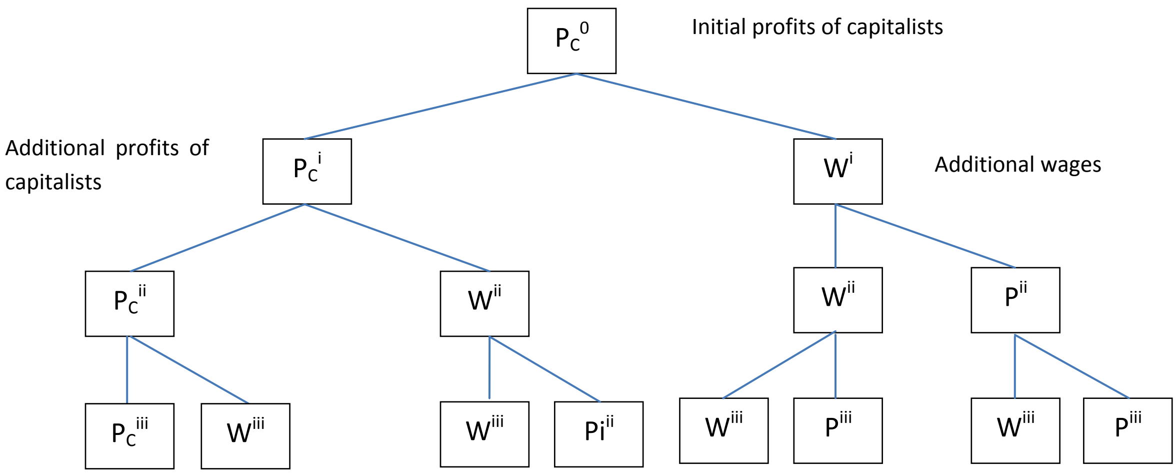
The additional income created by investment, i.e. $c_E(1 - r)Y^0$, is then split into wages and profits, whereby in the case of employees' investments the resulting profits belong solely to employees themselves. These additional wages and profits are again either consumed or invested, so that additional income is created, and so on. The rationale of the cumulative process out of employees' initial profits is the same as in the case of wages.

Regarding initial profits of capitalists Y_C^0, they are also firstly split into consumption and investment, according to the capitalists' propensities to consume b_c and invest c_c. The additional income created by capitalists' investment is split into profits and wages, whereby the profits pertain to the capitalists, yet the wages to employees. Part of these wages is also invested by employees, setting of an cumulative process equal to that of the initial wages. But also part of the capitalists' profits is again invested, thus creating further income, which again is split into profits and wages, with additional cumulative processes. The complexity of these processes is illustrated in Graph 1 and 2.

Graph 1. Additional income: wages



Graph 2. Additional income: profits



Using this and the basic logic of the short run cumulative process in the case of aggregate income, the outcome of the short run cumulative process for employees' and capitalists' income can be derived. Hence, the total income of employees, i.e. their total wage and profit income, and the end of the period is given by[5]:

$$Y_E^1 = \frac{Y^0}{1-c_E}\left(1 - \frac{r\alpha(1-c_C)}{(1-c_C r)}\right) \tag{63}$$

For simplicity the remainder of the analysis sets $v = 1$, which facilitates the presentation of results.

The income of the capitalists is given by:

$$Y_C^1 = \frac{\alpha r Y^0}{1-c_C r} \tag{64}$$

Adding the employees' and the capitalists' income results in the aggregate income of the economy. It is given by:

$$Y^1 = \frac{Y^0}{1-c_E}\left(1 + \frac{\alpha r(c_C - c_E)}{(1-c_C r)}\right) \tag{65}$$

[5] The full derivation is given in the Appendix.

Dynamics

For the analysis of the long run dynamics of income growth and distribution it is worthwhile to recapitulate that in the model the development of aggregate income over time is determined by four variables: the propensity to invest of employees c_E and capitalists c_c the profit rate r and the share of capitalists' profits in total profits α.

Notably, the first three variables are considered to be behavioural variables. Thus, in principal, they may be either exogenously or endogenously determined in the model, in the first case by assuming certain specific values for each variable, in the second case by assuming some function that relates them e.g. to income growth.

By contrast, α is considered to be strictly endogenous, as it is itself determined by the other three variables and thus changes, if one of these changes. This is because α is not only the share of capitalists' profits in total profits but at the same time also the capitalists' share in the total capital stock, and hence indicating the distribution of capital-ownership. This however may change over time if investments of employees and capitalists are not proportional to the distribution of the existing capital stock. At the same time α also determines aggregate income, so that in order to analyse the long run income dynamics the dynamics of α have to be clear first.

For this it is useful to start with the conditions under which α is stable. Formally, stability of α requires $\alpha_t = \alpha_{t+1}$ for any t in time. Using the short run model above, α_t is set to α_0, i.e. the capitalists' share in total profits at period zero. That is:

$$\alpha_0 = \frac{\alpha_0 r Y^0}{r Y^0} = \alpha_0 \tag{66}$$

Consequently α_{t+1} equals α_1, which is given by (see Appendix):

$$\alpha_1 = \frac{\alpha_0(1-c_E)}{(1-c_C r)+r\alpha_0(c_C-c_E)} \tag{67}$$

From this it follows that $\alpha_1 = \alpha_0$ has two solutions, the trivial solution being $\alpha_1 = \alpha_0 = 0$, while the second solution is:

$$\alpha = \frac{c_c r - c_E}{r(c_c - c_E)} \tag{68}$$

As α by definition only can take values from 0 to 1, the second solution requires for α to be positive: $c_c r > c_E$. If this is not given, it can be shown (see Appendix) that α converges to zero. Economically it is clear, because if $c_E > c_c r$ employees' invest more than the capitalists, so that the employees' share in the capital stock and hence profits increases over time, while the respective share of the capitalists becomes negligible small (and in the limit zero).

The movement of α towards the upper or lower bound has implications for the long run development of aggregate income. If α goes to zero, long run income will develop according to:

$$Y_{t+1} = Y_t \frac{1}{1-c_E} \tag{69}$$

This is given by setting α to zero in equation (XX) above. Hence in this case long run development will depend on the employees' investment propensity only. Therefore long run growth is given by:

$$Y_n = \frac{Y_0}{(1-c_E)^n} \tag{70}$$

Notably, because α is zero, profits and hence income of capitalists will go towards zero, so that aggregate income is equivalent with employees' income.

In case α moves to the upper bound long run income develops according to:

$$Y_{t+1} = Y_t \frac{1}{1-c_C r} \tag{71}$$

This equation shows that in the long run is dependent on the capitalists' investment propensity and on the distribution of income between wages and profits, given by the profit rate r. Yet, it is independent of the employees' investment propensity, which only affects the distribution of income between income of employees and capitalists. Accordingly, over time aggregate income will develop as:

$$Y_n = \frac{Y_0}{(1-c_C r)^n} \tag{72}$$

Capitalists' profits and hence income develops in the long run according to:

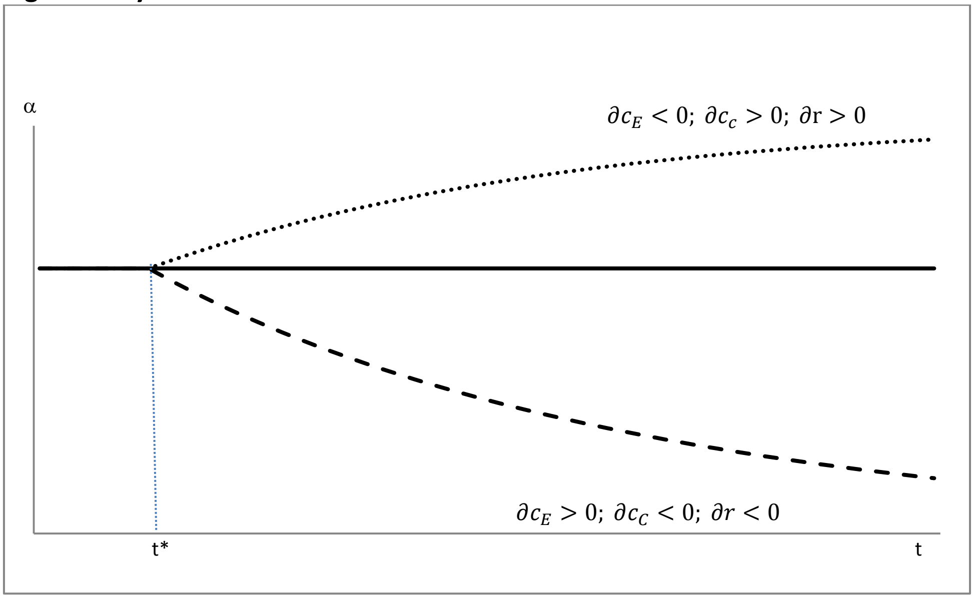
$$Y_C = r\alpha \frac{Y_0}{(1-c_C r)^n} = \frac{c_C r - c_E}{c_C - c_E} \frac{Y_0}{(1-c_C r)^n} \tag{73}$$

In turn, employees' income is given by

$$Y_E = \frac{c_C}{c_C - c_E} \frac{(1-r)Y_0}{(1-c_C r)^n} \tag{74}$$

To illustrate the dynamic properties, Figure 1 shows the development of α, given ceteris paribus changes in the profit rate as well as the investment propensity of employees and capitalists, respectively. Thereby the straight line represents a stable development of α over time, i.e. the distribution of total income between employees and capitalists does not change over time, as c_E, c_C and r are constant. At point t* changes to these three variables are introduced, so that a moves away from its initial value. The upper line represents an increase in α caused either by a decrease in the employees' investment propensity or an increase of capitalist investment or of the profit rate. That is the capitalists' share in profits and capital stock increases over time, as a) employees invest less than before t*, so that ceteris paribus there share in the capital stock decreases over time; b) capitalists invest more than before t*, hence, their share in the capital stock increases; c) the increase of the profit rate shifts income from employees to capitalists. As they are assumed to have a higher investment propensity, their investment is increased, while at the same employees investment is decreased. Consequently α increases over time and converges to a new stable value according to equation (68). The lower line represents a decrease in α, as, ceteris paribus, either c_E increases, or c_C and r decrease.

Figure 1. Dynamics of α



According to the development of α over time, the distribution of aggregate income between employees and capitalists turns in favour of one the two groups depending on the changes in the investment propensities and the profit rate. To illustrate this, it is assumed that aggregate income shifts away from capitalists to employees. This means analysing the cases where there is an increase in the employees' investment propensity, a decrease in the capitalists' investment propensity or a reduction of the profit rate (Figure 2).

Given these changes, the ratio of employees' income to capitalists' income, which serves as a rough inequality measure, shows some differences in the adjustment process. In the case of an increasing employees' investment or decreasing capitalists' investment, the income ratio increase smoothly over time. In the case of a decrease in the profit rate, there is an initial, abrupt change, and after that the ratio increases smoothly, too. Thereby, the extent to which the ratio shifts depends on how much the investment propensities and the profit rate shift. In the example below, the increase in the employees' investment propensities has the biggest redistribution effect. However, this is because of how the underlying values for the variables have been chosen to calibrate the model. By, choosing different values, the changes in the profit rate or the decline in the capitalists' investment may have bigger effects.

Figure 2. Ratio of employees' income to capitalists' income

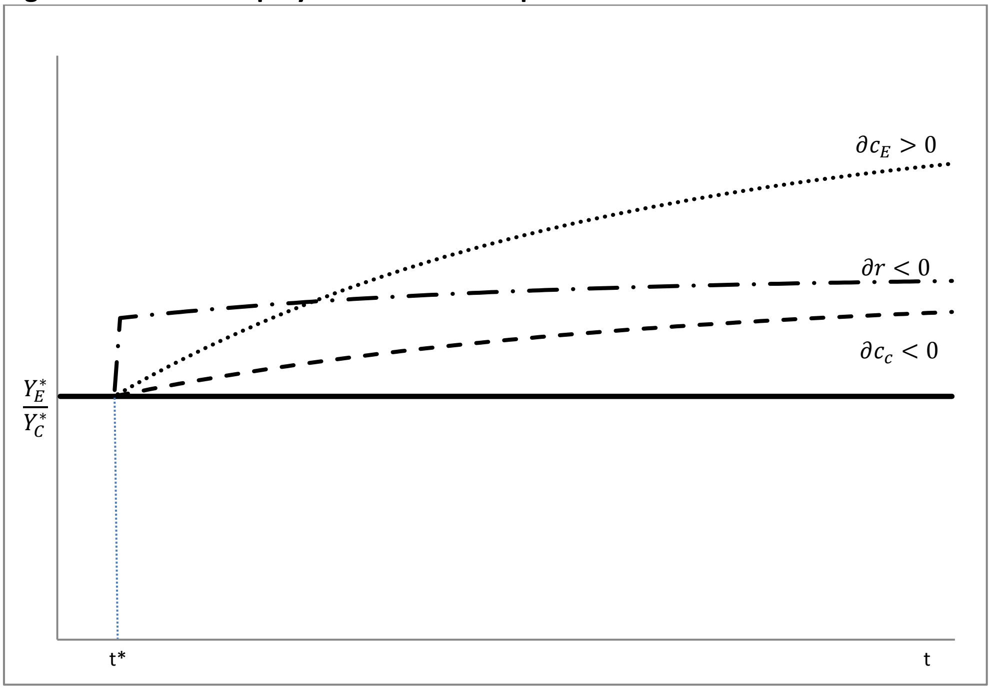


Figure 3 shows the dynamics of aggregate income. Clearly, an increase in c_E has a positive effect on long run income development, while a decrease in c_c and in the profit rate r, reduces aggregate income compared to the base line. In the case of the profit rate, income is lowered because, part of aggregate income is shifted to the group that has a lower investment propensity, which reduces aggregate investment and hence aggregate income.

Figure 3. Dynamics of aggregate income

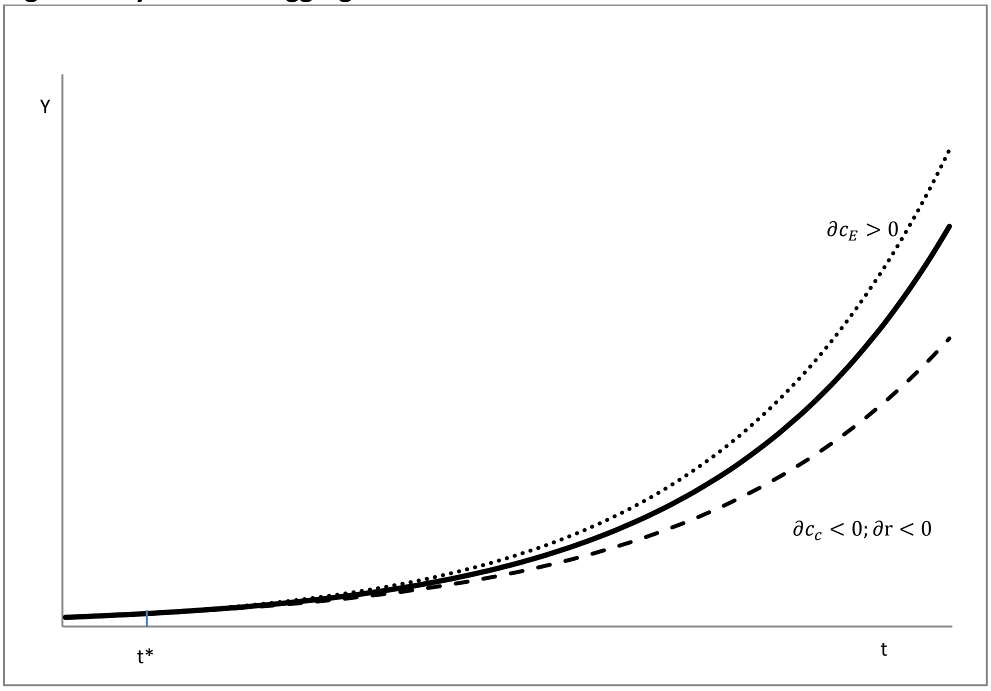
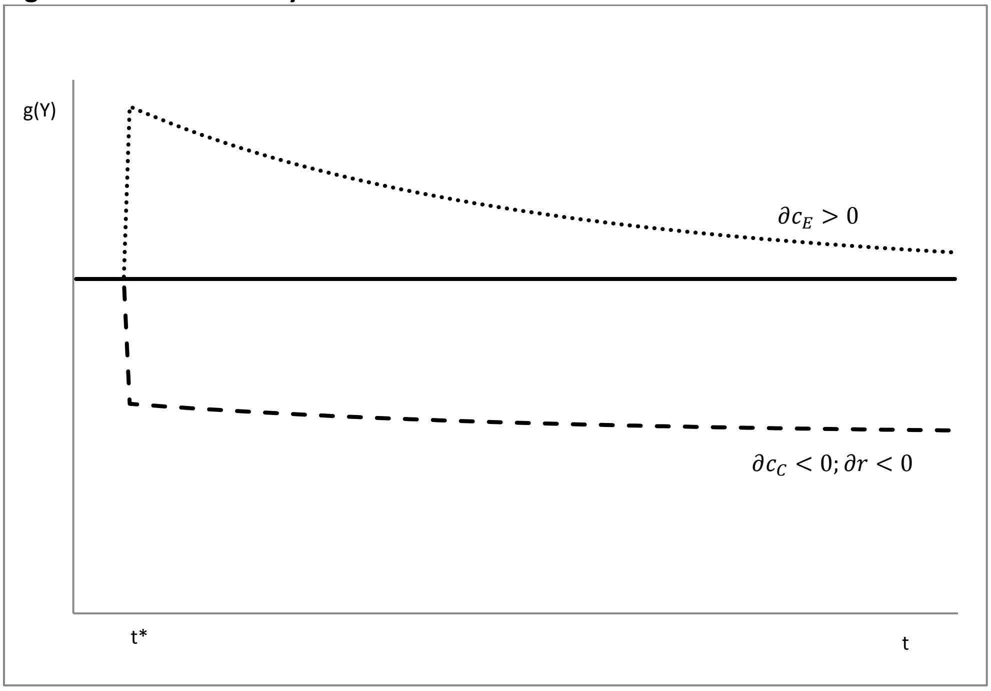


Figure 4 illustrates the dynamics of the aggregate income growth rate. Hence, the increase in c_E, as well as the decrease in c_c and r have immediate effects on the period to period growth rate, as it either increases or declines depending on the sign of the changes. However it is also shown, that though an increase in c_E has an immediate effect on the growth rate, this effect diminishes over time, as the growth rate returns to its initial, base line level (notably this holds for $c_C r > c_E$). In case of a decline of c_c and r the growth rate drops immediately and then converges towards a new stable value, which is lower than the base line growth rate.

Figure 4. Growth rate dynamics

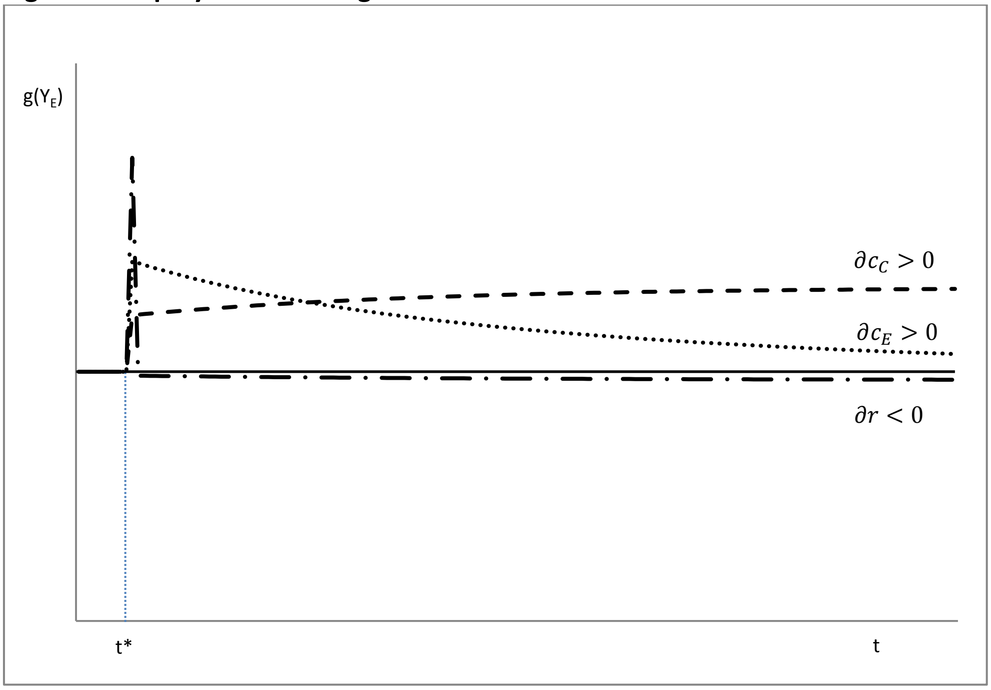


The example above was concerned with the effects of redistribution of income from capitalists to employees. It shows that the only way for redistribution to not decrease aggregate income and its growth is via an increase of the employees' investment propensity. The other two options lead to a redistribution of income, yet at the cost of lower aggregate income.

If the concern is only about an increase in employees' income one may compare the effects of an increase in c_E and c_c, as the latter increases not only aggregate and capitalists' income but also employees' income. Figure 5 illustrates this via the growth rate of employees' income, adding for comparison reasons also the effect caused by a decrease in the profit rate.

It shows, that even if in the short run employees might benefit more by an increase of their investment, in the long run the may benefit more (in terms of absolute income), by an increase of the capitalists' investment, as the latter increases the growth rate of income permanently, while an increase in c_E does so only temporarily. However this comes at the cost of a shift of the income distribution in favour of the capitalists. The positive effects of a decline in the profit rate are short lived. There may be a sizeable one time effect, yet the negative effect on long run aggregate growth compensates this effect relatively quickly.

Figure 5. Employees' income growth rate



A note on convergence

Closely connected to aggregate income growth is the issue of income convergence. It can be described as the question whether and under which conditions poor economies will catch up in income terms with more prosperous economies. Looking at it in a stylised manner, the problem for a country i to converge income terms to country j, has at least two aspects, as aggregate income is defined in 'real' terms via the demand side and in potential terms via the supply side.

The starting point for a convergence process is characterised by:

$$Y_i < Y_j;\ Y_i^* < Y_j^*$$

Hence, both real and potential aggregate income are lower in country i than in country j[6]. This holds for situation, where the capacity utilisation u_i in economy i is either equal or lower than capacity utilisation in country j. Indeed, it may also be assumed that, as a tendency the poorer country also has a lower rate of capacity utilisation, so that one additional starting point may be added, namely:

$$u_i < u_j$$

Convergence therefore requires:

$$g(Y_i) > g(Y_j);\ g(Y_i^*) > g(Y_j^*);\ g(u_i) > g(u_j)$$

Hence, convergence means, that both the demands side as well as the supply side, potential income of the poorer country i have to grow faster than the real and potential income of country j. Additionally it can also be required that capacity utilisation grows faster. In the model above it has been shown that the growth of all three elements is a function of the investment rate and the

[6] Usually convergence is expressed in terms of income per head. In the stylised view on convergence, it is, for simplicity, assumed that both economies have the same, constant number of population.

capital-output ratio, i.e. the level of technology. From that, necessary conditions for convergence can be derived. This is illustrated by analysing the speed of convergence.

Using the exponential growth of aggregate income as approximation, the speed of convergence between two countries can be determined. Assuming an initial situation where the income of country i Y_i is lower than country j's income Y_j, Y_i can be given as a fraction α of Y_j:

$$Y_{i,0} = \alpha_0 Y_{j,0}$$

Hence in period one the difference between country i and j is represented by:

$$\alpha_1 = \frac{Y_{i,1}}{Y_{j,1}} = \frac{Y_{i,0} e^{\frac{c_i}{v_i - c_i}}}{Y_{j,0} e^{\frac{c_j}{v_j - c_j}}}$$

Accordingly the development path of the difference between the two countries is:

$$\alpha_n = \frac{Y_{i,0} e^{\frac{c_i}{v_i - c_i} n}}{Y_{j,0} e^{\frac{c_j}{v_j - c_j} n}} = \alpha_0 e^{\left(\frac{c_i}{v_i - c_i} - \frac{c_j}{v_j - c_j}\right) n}$$

Since an equality of both countries' incomes requires $\alpha_n = 1$, taking (natural) logarithm and solving for n, gives the approximate time required for convergence:

$$n = \frac{\ln \alpha_0}{\frac{c_j}{v_j - c_j} - \frac{c_i}{v_i - c_i}}$$

As country's i income Y_i was assumed to be lower than country's j income Y_j, $\alpha_0$is lower than one and hence the natural logarithm negative. Hence, for convergence it requires the denominator in XX also to be negative. From this the necessary condition for convergence can be derived (see Appendix):

$$c_i / v_i > c_j / v_j$$

Accordingly, if both countries have the same level of technology, the investment rate of country i c_i has to be higher than that of country j. However, above equation also shows that convergence might fail, even if country i's investment rate is higher than country j's if its level of technology is too low, i.e. v_i is too high. Rewriting above equation illustrates this:

$$c_i / c_j > v_i / v_j$$

Hence, for convergence the share of investment of country i, compared to country j, has to be higher the higher the difference in technology is. Vice versa, convergence can also occur if the investment ration of country i is lower, given that it has a more advanced level of technology than country j, i.e. $v_i < v_j$.

Concluding remarks

To this point the model presented in this paper deals with a closed economy without a government sector. Creating an open economy variant of the model probably is a straightforward exercise, at least as far as the basic model is concerned. Introducing foreign activities, i.e. exports (X) and imports (M) is simple, though importantly the starting point of the short run cumulative process is expected to look like:

$$Y_0 + M_0 = C_0 + I_0 + X_0 + S_0$$

as the initial income is augmented by imports, which both can be used for consumption, investments, savings or exports. Thereby the trade deficit or surplus $X_0 - M_0$ can be regarded as foreign savings (or borrowing if negative), which, just as domestic savings, accumulate over time. Assuming a trade deficit and hence a growing stock of foreign debt and introducing an interest rate on this debt, such analysis could be expanded to issues of the external stability of an economy. Nevertheless, this is left for future work.

Including a government sector into the framework of a double cumulative development process appears to be more difficult. It may be relatively simple in the basic model, where distributional aspects are absent. Yet, many of the government's activities have, either directly or indirectly, effects on the distribution of income, and, following the logic of the model, thus consequences for economic growth and development. Including the government requires a clear definition of the state's role in the economy, which may not be easy to find. Indeed, assumptions would have to be made regarding the distributional effects of government activities, i.e. to whom the income created by the state's consumption and investment belongs and who finances the states expenditures. This immediately leads to issues of government borrowing and debt and again the distributional effects thereof. Implementing this formally includes the introduction of a couple more variables in the dynamic setting of the model, which both, from an analytical point of view and presentation wise is not an easy task.

It was said in the beginning that the analysis focusses on the basic macroeconomic mechanisms of income growth and distribution. Thus, the paper was neither overly concerned about the particular size of the behavioural variables, nor did it pay much attention on whether they might be endogenously determined. For analytical reasons, they were assumed to be exogenous, though in the back of the mind a number of possibilities to endogenise some variables were considered but not analysed in full detail in order not to lose the focus. One promising idea centred around the relation between investment and savings, based on the conjecture that, if savings are not identical to investment, from a macroeconomic point of view the existence of savings might help to explain the recurrent phenomenon of bubbles and financial and economic crisis. A second idea concerns the connection of income distribution and economic development. By making the profit rate as well as the investment rates endogenous it should be easily possible to derive a number potential scenarios helping to explain real world problems, e.g. like a growth slowdown in developing countries that may be due to a redistribution of income from capitalists to employees (e.g. as with development there bargaining power increases), which potentially decreases the average investment rate and hence growth.

Appendix

Quasi investment multiplier

The Keynesian investment multiplier measures the change in GDP given a one unit increase in investment. Applying this to the cumulative model, investment is given by:

$$I_{i,0}^{M} = I_{i,0} + 1 = cY_{i,0} + 1 = Y_{i,0}\left(c + {}^{1}/_{Y_{i,0}}\right)$$

Whereby the superscript M denotes investment in the multiplier case, and $c + 1/Y_{i,0} = c^{M}$.

From above it can be derived that GDP in case of a one unit increase in investment is then given by:

$$Y_{i,0}^{f,M} = \frac{Y_{i,0}}{(1 - c^{M}/v)}$$

The difference between the aggregate multiplier GDP $Y_{i,0}^{f,M}$ and the non-multiplier GDP $Y_{i,0}^{f}$ is then

$$Y_{i,0}^{f,M} - Y_{i,0}^{f} = \frac{Y_{i,0}}{(1 - c^{M}/v)} - \frac{Y_{i,0}}{(1 - {}^{c}/_{v})} = dY$$

Since the difference between $I_{i,0}$ and $I_{i,0}^{M}$ was defined to be one above equation reduces to:

$$dY = \frac{1/v}{(1 - c^{M}/v)(1 - {}^{c}/_{v})}$$

For small changes the investment multiplier m of this model might be approximated by:

$$\frac{dY}{dI} = \frac{1}{v(1 - {}^{c}/_{v})^{2}}$$

Cumulative processes in the case of income distribution

Employee wages are given by:

$$Y_{E,W}^{0} = (1 - r)Y^{0}$$

In the absence of savings this income is split between consumption and investment:

$$Y_{E,W}^{0} = b_{E}(1 - r)Y^{0} + c_{E}(1 - r)Y^{0}$$

Hence assuming $v = 1$ first round investments and hence first round income equals:

$$I_{E,W}^{i} = c_{E}(1 - r)Y^{0} = Y_{E,W}^{i}$$

This first round income consists of wages and profits, so that it may also be written as:

$$Y_{E,W}^{i} = (1 - r)Y_{E,W}^{i} + rY_{E,W}^{i}$$

In principle, both the wages and the profits are split again into consumption and investment, thus generating additional income from investment out wages and out of profits. This additional incomes would again be split, firstly into wages and profits and secondly into consumption and investment thereof and so on. In the limit this would result in 2^n terms for wages and profits, if wages and profits of employees (out of employees' wages) are to be calculated separately.

However, since both wages and profits pertain to employees, a convenient shortcut is to calculate total accumulated employees' income out wages first and then split it into wages and profits. In this case total income out of wages is given by (in analogy to the calculation of aggregate income):

$$Y_{E,W}^{1} = \frac{(1-r)Y^{0}}{1-c_E}$$

The wage part thereof is then given by:

$$Y_{E,Ww}^{1} = (1-r)Y^{0}\left(\frac{1-rc_E}{1-c_E}\right)$$

The profit part is: Profits

$$Y_{E,Wp}^{1} = (1-r)Y^{0}\left(\frac{rc_E}{1-c_E}\right)$$

This is due to the fact that the actual starting income for calculating profits is: $c_E(1-r)Y^{0}$ and not $(1-r)Y^{0}$.

In similar fashion the short-run accumulated income out of employees' profits can be calculated. Employees' profits are given by:

$$Y_{E,P}^{0} = (1-\alpha)rY^{0}$$

So that first round income is:

$$Y_{E,P}^{i} = c_E(1-\alpha)rY^{0}$$

This is split into wages and profits, which lead to consumption and investment and hence additional income etc. In the end total employees income out of profits is given by:

$$Y_{E,P}^{1} = \frac{(1-\alpha)rY^{0}}{1-c_E}$$

The wage part therein is:

$$Y_{E,Pw}^{1} = (1-r)(1-\alpha)rY^{0}\left(\frac{c_E}{1-c_E}\right)$$

Again, the starting income for the derivation of wages is $c_E(1-\alpha)rY^{0}$ and not $(1-\alpha)rY^{0}$.

Correspondingly total employee profits (out of their initial profits) are given by:

$$Y_{E,Pp}^{1} = r(1-\alpha)Y^{0}\left(1+\frac{rc_E}{1-c_E}\right)$$

The short rum cumulative triggered by capitalist profits is a bit more difficult to develop. Capitalists' profits are given by:

$$Y_{C}^{0} = \alpha rY^{0}$$

Again, in the absence of savings, part of it is consumed part of it invested, so that:

$$Y_C^0 = b_c \alpha r Y^0 + c_c \alpha r Y^0$$

First round income is then:

$$Y_C^i = c_c \alpha r Y^0$$

This income however is split into wages, as capitalists have to pay the employees, and profits pertaining to the capitalists.

$$Y_C^i = (1-r) c_c \alpha r Y^0 + r c_c \alpha r Y^0$$

The wages of the employees are themselves again the starting point for a cumulative process, just like with pure employee wages above. Hence one could evaluate the employees' income out of this first round income, yet this would have to be done again for the second round income as capitalists also invest part of their additional income. This would lead to further income, which again would be split into wages and capitalists' profits. These second round wages would be the starting point of a second cumulative process out wages. The same occurs for all following rounds.

To get the total employees income out of this cumulative process it is easier to first check how the starting income of the employees develops as the capitalists invest. Hence in the first round the starting income for employees is given by

$$Y_{C,E}^i = (1-r) c_c \alpha r Y^0$$

The second round starting income of the employees is:

$$Y_{C,E}^{ii} = (1-r) r c_C^2 \alpha r Y^0$$

In the third round it is:

$$Y_{C,E}^{iii} = (1-r) r^2 c_C^3 \alpha r Y^0$$

and so on.

In the limit total starting income of the employees can therefore be given as

$$Y_{C,E}^0 = \frac{(1-r) c_c \alpha r Y^0}{1 - c_c r}$$

Correspondingly through the cumulative process the total income of employees (out of capitalists' profits) amounts to

$$Y_{C,E}^1 = \frac{(1-r) c_c \alpha r Y^0}{(1 - c_c r)(1 - c_E)}$$

The wage part thereof is (in analogy to employees' wages):

$$Y_{C,Ew}^1 = (1-r) r c_c \alpha Y^0 \left(\frac{1 - r c_E}{(1 - c_c r)(1 - c_E)} \right)$$

The employees' profit part is then:

$$Y_{C,Ep}^{1} = (1-r)rc_c\alpha Y^0\left(\frac{rc_E}{(1-c_c r)(1-c_E)}\right)$$

The capitalists' profits themselves develop more regularly in the cumulative process, as the first round additional income for capitalists is given by:

$$Y_C^i = rc_c\alpha rY^0$$

In the second round it is:

$$Y_C^{ii} = r^2c_c^2\alpha rY^0$$

So that over n-rounds is equal to:

$$Y_C^1 = \alpha rY^0 + rc_c\alpha rY^0(1 + c_c r + c_c^2r^2 + \cdots + c_c^{n-1}r^{n-1})$$

Hence in the limit, the total income of capitalists through the cumulative process is:

$$Y_C^1 = \frac{\alpha rY^0}{1-c_c r}$$

Adding the various elements shows that total wages are given by

$$Y_W^1 = Y_{E,Ww}^1 + Y_{E,Pw}^1 + Y_{C,Ew}^1$$

or:

$$Y_W^1 = (1-r)Y^0\left[\frac{1}{1-c_E} + \frac{\alpha r(c_c - c_E)}{(1-c_c r)(1-c_E)}\right]$$

Similar total employee profits are:

$$Y_{EP}^1 = Y_{E,Wp}^1 + Y_{E,Pp}^1 + Y_{C,Ep}^1$$

or:

$$Y_{EP}^1 = rY^0\left[\frac{1-\alpha}{1-c_E} + \frac{\alpha(1-r)c_E}{(1-c_c r)(1-c_E)}\right]$$

Total profits in the economy are:

$$Y_P^1 = Y_C^1 + Y_{EP}^1$$

$$Y_P^1 = rY^0\left[\frac{1}{1-c_E} + \frac{\alpha r(c_c - c_E)}{(1-c_c r)(1-c_E)}\right]$$

Furthermore total income of employees is:

$$Y_E^1 = Y_W^1 + Y_{EP}^1$$

$$Y_E^1 = \frac{Y^0}{1-c_E}\left(1 - \frac{r\alpha(1-c_c)}{(1-c_c r)}\right)$$

Finally, aggregate income is given by:

$$Y^1 = Y_E^1 + Y_C^1$$

This is:

$$Y^1 = \frac{Y^0}{1 - c_E}\left(1 + \frac{\alpha r(c_c - c_E)}{(1 - c_c r)}\right)$$

Condition for convergence

The approximate time required for convergence is given by:

$$n = \frac{\ln \alpha_0}{\frac{c_j}{v_j - c_j} - \frac{c_i}{v_i - c_i}}$$

Thereby α_0 is the initial gap in income between country i and country j, i.e. $Y_{i,0} < Y_{j,0}$ and $Y_{i,0} = \alpha_0 Y_{j,0}$. Hence the natural logarithm of α_0 is negative. For convergence it thus requires the denominator also to be negative:

$$\frac{c_i}{v_i - c_i} > \frac{c_j}{v_j - c_j}$$

Assuming that c_i is c_j times a factor, i.e. $c_i = bc_j$, the above inequality solves:

$$b > \frac{v_i}{v_j}$$

Substituting for b gives the necessary condition for convergence:

$$c_i/v_i > c_j/v_j$$